The Nonnegative Rank Factorizations of Nonnegative Matrices

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ABSTRACT

Let $A \in P_r^{m \times n}$, the set of all $m \times n$ nonnegative matrices having the same rank r. For matrices A in $P_n^{m \times n}$, we introduce the concepts of "A has only trivial nonnegative rank factorizations" and "A can have nontrivial nonnegative rank factorizations." Correspondingly, the set $P_n^{m \times n}$ is divided into two disjoint subsets $P_{(1)}$ and $P_{(2)}$ such that $P_{(1)} \cup P_{(2)} = P_n^{m \times n}$. It happens that the concept of "A has only trivial nonnegative rank factorizations" is a generalization of "A is prime in $P_n^{n \times n}$." We characterize the sets $P_{(1)}$ and $P_{(2)}$. Some of our results generalize some theorems in the paper of Daniel J. Richman and Hans Schneider [9].

1. INTRODUCTION

Most of the notation and definitions in this paper are standard and agree with those in the references.

Let $A \in P_r^{m \times n}$, the set of all $m \times n$ nonnegative matrices having the same rank r. A is said to have a nonnegative rank factorization (n.r.f. for short) if there exist matrices $B \in P_r^{m \times r}$ and $C \in P_r^{r \times n}$ such that $A = BC$. Not every nonnegative matrix need have an n.r.f. [1]. L. B. Thomas [1], S. L. Campbell and G. D. Poole [2], and Bit-Shun Tam [3] gave necessary and sufficient conditions for A to have an n.r.f. Some of the conditions in the above literatures are essentially the same. If $r = \min\{m, n\}$, then there exists a trivial n.r.f. of A involving the identity I_r. Suppose r satisfies $0 < r < \min\{m, n\}$. Without loss of generality, we may assume that A is already

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partitioned as

$$A = \begin{pmatrix} M & MG \\ FM & FMG \end{pmatrix}$$

where M is $r \times r$ and nonsingular.

Now, we can restate their results in the following form:

THEOREM 1.1. *Suppose $A \in P_r^{m \times n}$, $0 < r < \min\{m, n\}$, and A is partitioned in the form given above. Then the following are equivalent:*

(i) *A has an n.r.f.*

(ii) *There exists a set M_1 of nonnegative and linearly independent vectors in R^n such that the cone $C_r(A)$ (the cone generated by the rows of A) is contained in the cone generated by M_1.*

(iii) *There exists an $r \times r$ nonsingular matrix N such that $[M, MG]R_+^n \subseteq NR_+^r \subseteq \operatorname{Im} F \cap R_+^r$.*

(iv) *A is expressible as $x_1 y_1^T + \cdots + x_r y_r^T$ where $x_i \in R_+^m$ and $y_i \in R_+^n$, $i = 1, \ldots, r,$.*

(v) *There exists an (r-dimensional) simplicial cone K such that $AR_+^n \subseteq K \subseteq \operatorname{Im} A \cap R_+^m$.*

Tam also pointed out in [3] that when the conditions are satisfied, the representation of A given in (iv) is unique up to the order of its summands if and only if the simplicial cone K which satisfies (v) is unique. The term "the dimension of a cone" has different meanings in [2] and [3]. By "the dimension of a cone," Campbell and Poole mean the number of extreme vectors of the cone, while Tam means the dimension of its linear span. In this paper, as long as we confine ourselves to full-column-rank matrices, these two meanings are equivalent. From Theorem 1.1, it is obvious that if $A = BC$ is an n.r.f. of A, then corresponding to this special factorization, A can be expressed as a sum of r nonnegative rank-one matrices in essentially one way and also there exists a special (r-dimensional) simplicial cone K which satisfies (v). Whether a nonnegative matrix A may have two (or more) essentially different representations of the form given in (iv) [or equivalently, there exist two (or more) different r-dimensional simplicial cones that satisfy the cone containment relation in (v)] is not clear. This problem may suitably be called the nonuniqueness of the n.r.f. of a nonnegative matrix and may have some connections with the so-called cone-containment problems posed by G. D. Poole and S. L. Campbell in the South Carolina Mini Matrix Conference [4].

For any matrix $A \in P_n^{m \times n}$, let us introduce the concepts of "A has only trivial n.r.f." and "A can have nontrivial n.r.f." Throughout Sections 2 and 3, we will assume that $A \in P_n^{m \times n}$. Let M_n be any nonnegative monomial of order n, that is, M_n has exactly one positive entry in each row and in each column.

DEFINITION 1.2. BC is a trivial n.r.f. of A if there is some monomial M_n such that $B = AM_n^{-1}$, $C = M_n$, and

$$A \neq \begin{pmatrix} M_n \\ 0 \end{pmatrix}. \tag{1.2}$$

It is obvious that any $A \in P_n^{m \times n}$ can have a trivial n.r.f. If A is not of the form (1.2) and A has an n.r.f. $A = BC$ different from a trivial n.r.f., we will say that A has a nontrivial n.r.f.

Since a singular prime in $P^{n \times n}$ has no n.r.f. [9], the concept "A has only trivial n.r.f." is equivalent to "A is prime" whenever $m = n$ [9]. Thus, if $P_{(1)} \subseteq P_n^{m \times n}$ is the subset of $P_n^{m \times n}$ such that $A_1 \in P_{(1)}$ implies that A_1 has only trivial n.r.f., and $P_{(2)} \subseteq P_n^{m \times n}$ is the subset of $P_n^{m \times n}$ such that $A_2 \in P_{(2)}$ implies that A_2 does have a nontrivial n.r.f., then we have $P_n^{m \times n} = P_{(1)} \cup P_{(2)}$ and $P_{(1)} \cap P_{(2)} = \varnothing$, the empty set. In this paper, we try to characterize these two subsets of $P_n^{m \times n}$. Some of our results are generalizations of factorization theorems (Theorems 2.4, 2.6, 2.7, 2.8 of [9]) of Daniel J. Richman and Hans Schneider.

2. SOME BASIC RESULTS

Suppose now that $A = BC$ is a trivial n.r.f. of A. Then in Theorem 1.1, we have $N = MM_n^{-1}$ in (iii), and the $r = n$-dimensional simplicial cone K that satisfies (v) is generated by the column vectors of A (or their positive scalar multiples) in any order. Since two simplicial cones having the same number of extreme vectors are identified if and only if they have the same set of extreme vectors (up to a positive scalar multiple) and A and B both have full column rank, it is immediate that $A = BC$ is a trivial n.r.f. if and only if the cone-containment relation $AR_+^n \subseteq K$ $(= BR_+^n) \subseteq \operatorname{Im} A \cap R_+^m$ holds. That A has full column rank implies $\operatorname{Im} A \cap R_+^m = R_+^m$. On the other hand, the simplicial cone R_+^m is generated by the columns of an $m \times n$ matrix of the following form:

$$\begin{pmatrix} D \\ 0 \end{pmatrix},$$

where D is a nonsingular nonnegative diagonal matrix. Therefore, K ($= BR_+^n$) $= \operatorname{Im} A \cap R_+^m$ holds if and only if A is of the form (1.2). Thus, we have come to the following conclusion.

THEOREM 2.1. *Every $A \in P_n^{m \times n}$ has a trivial n.r.f. For any such n.r.f., the corresponding $r = n$-dimensional simplicial cone K satisfies the cone-containment relation $AR_+^n = K \subseteq \operatorname{Im} A \cap R_+^m$. Furthermore, $AR_+^n = K = \operatorname{Im} A \cap R_+^m$ holds if and only if A is of the form* (1.2).

COROLLARY 2.1. *A nonnegative weakly monotone matrix of full column rank has and can only have a trivial n.r.f.*

This can be seen by noticing that an equivalent condition for an $m \times n$ nonnegative matrix A to be weakly monotone is that $AR_+^n = \operatorname{Im} A \cap R_+^m$ [7]. Let $A \in P_n^{m \times n}$. Assume that A is not of the form (1.2). Combining Theorem 1.1 and Theorem 2.1, we obtain a necessary and sufficient condition for A to have nontrivial n.r.f. which may be described as follows.

THEOREM 2.2. *A has a nontrivial n.r.f. if and only if there exists an $r = n$-dimensional simplicial cone K satisfying the following strict cone-containment relation: $AR_+^n \subset K \subset \operatorname{Im} A \cap R_+^m$.*

3. THE NONTRIVIAL n.r.f. OF A (THE CHARACTERIZATION OF $P_{(2)}$)

THEOREM 3.1. *Let $A \in P_2^{2 \times 2}$. Assume that A is not a monomial. Then $A \in P_{(2)}$. Moreover, A has in fact infinitely many nontrivial n.r.f.'s.*

Proof. Without loss of generality, we may assume that the determinant $\det(A) > 0$, since A has an n.r.f. if and only if AP_2 has an n.r.f., where P_2 is a suitable permutation matrix of order 2. For any positive scalar α, define $B_\alpha = A + \alpha I_2$. Then $B_\alpha \in P^{2 \times 2}$. Whenever $-\alpha \notin \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A, by direct computation we get $B_\alpha^{-1} A \in P_2^{2 \times 2}$.

Therefore, $A = B_\alpha C_\alpha$ is a nontrivial n.r.f. of A for each $\alpha > 0$ and $-\alpha \notin \sigma(A)$, where $C_\alpha = B_\alpha^{-1} A$. ∎

Theorem 3.1 again confirms that there are no primes in N_2 [9].

THEOREM 3.2. *Let $A \in P_n^{n \times n}$ be a positive matrix, i.e. $A = (a_{ij})$, with $a_{ij} > 0$, $i, j = 1, 2, \ldots, n$. Then $A \in P_{(2)}$. Moreover, A has in fact infinitely many nontrivial n.r.f.'s.*

Proof. Let $B_\alpha = A + \alpha I_n$, $C_\alpha = B_\alpha^{-1}A$, where α is any positive scalar such that $-\alpha \notin \sigma(A)$. The determinant of the matrix B_α is a monic polynomial in α of degree n. Therefore, $\det(B_\alpha) > 0$ for sufficiently large scalar α. Let M_{ij} be the (i, j)-cofactor of B_α, i.e.,

$$M_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11}+\alpha & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn}+\alpha \end{vmatrix}.$$

We have

$$(\operatorname{adj}(B_\alpha)A)_{i,j} = \sum_{k=1}^{n} a_{kj} M_{ki}, \qquad i, j = 1, \dots, n,$$

where $(X)_{i,j}$ is the (i, j) element of a matrix X and $\operatorname{adj}(B_\alpha)$ denotes the adjoint matrix of B_α.

Note that M_{ii} is a monic polynomial in α of degree $n-1$ and M_{ki}, $k \neq i$, is a polynomial in α with degree no greater than $n-2$. Therefore, for each pair (i, j), there exists a positive scalar $\alpha_0(i, j)$, such that whenever $\alpha \geqslant \alpha_0(i, j)$, $(\operatorname{adj}(B_\alpha)A)_{i,j} \geqslant 0$ provided that $a_{ij} > 0$. The total number of possible (i, j) pairs is n^2. Taking $\bar{\alpha} = \max_{1 \leqslant i, j \leqslant n}\{\alpha_0(i, j)\}$, we see that if $a_{i,j} > 0$, $i, j = 1, \dots, n$, then whenever $\alpha \geqslant \bar{\alpha}$, we have $(\operatorname{adj}(B_\alpha)A)_{i,j} \geqslant 0$, $i, j = 1, \dots, n$. ■

For a matrix A, we define a_j to be its jth column. By A^* (the pattern of A), we denote the (0,1) matrix defined by $a^*_{i,j} = 1$ if $a_{ij} > 0$, and $a^*_{ij} = 0$ if $a_{ij} = 0$, where $A = (a_{ij}) \in P_n^{m \times n}$. We use the componentwise partial order on $P_n^{m \times n}$ and on the set of column n-tuples.

THEOREM 3.3. *Let $A \in P_n^{m \times n}$ with $n \geqslant 2$. Let $1 \leqslant i, k \leqslant n$ and $i \neq k$. If $a_i^* \geqslant a_k^*$, then A has nontrivial n.r.f.*

Note that Theorem 3.3 can be viewed as a generalization of Theorem 2.4 in [9].

Proof. Since for any permutation matrix P_n of order n, AP_n has nontrivial n.r.f. if and only if A has, we may assume without loss of generality that

$a_1^* \geqslant a_2^*$. Hence there exists a positive ε such that $b_1 = a_1 - a_2\varepsilon \geqslant 0$ and $b_1^* = a_1^*$. Let $b_i = a_i$, $i = 2,\dots,n$. Then $B = [b_1 b_2 \cdots b_n] \in P_n^{m\times n}$. For any nonnegative monomial M_n of order n, $B \neq AM_n$. Let

$$C = \begin{pmatrix} 1 & 0 & \\ \varepsilon & 1 & \\ & & I_{n-2} \end{pmatrix}$$

Then C is not a monomial. But we have that $A = BC$ is a nontrivial n.r.f. of A. ∎

The converse of Theorem 3.3 is false. A counterexample is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}.$$

4. THE TRIVIAL n.r.f. OF A (THE CHARACTERIZATION OF $P_{(1)}$)

DEFINITION 4.1. Let $A \in P_n^{m\times n}$. A is called fully indecomposable if there do not exist permutation matrices P and Q (of order m and n respectively) such that

$$PAQ = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} is a square matrix of order k with $0 < k < n$.

THEOREM 4.1. *Let $n \geqslant 2$ and let $A \in P_n^{m\times n}$. If*

(i) A is fully indecomposable, and
(ii) $(a_i^*)^T a_k^* \leqslant 1$ for all i, k such that $1 \leqslant i, k \leqslant n$ and $i \neq k$,

then A can have only trivial n.r.f.

Note that Theorem 4.1 is a generalization of Theorem 2.6 in [9]. By examining the proof of Theorem 2.6 in [9] carefully, we can find out that the logical inferences there can work well to prove Theorem 4.1 with only slight modifications. The proof is omitted.

THEOREM 4.2. *Let $A \in P_n^{m \times n}$, and let A have only trivial n.r.f. Then there exists an r, $1 \leqslant r \leqslant n$, and a fully indecomposable matrix $A_1 \in P_r^{(m-n+r)\times r}$ such that*

$$PAQ = \begin{pmatrix} A_1 & 0 \\ 0 & D \end{pmatrix}, \tag{4.1}$$

where P and Q are permutation matrices (of order m and n respectively), D is a nonsingular diagonal matrix in $P^{(n-r)\times(n-r)}$, and A_1 has only trivial n.r.f.

Note that Theorem 4.2 is a generalization of Theorem 2.7 in [9]. The proof is essentially the same, and we again omit it.

Theorem 4.3 is a generalization of Theorem 2.8 in [9].

THEOREM 4.3. *Let the two positive integers m, n be such that $m \geqslant n$, and let $A_1 \in P_r^{(m-n+r)\times r}$. Let M_{n-r} be a nonnegative monomial of order $n-r$, where $1 \leqslant r \leqslant n$.*

If A_1 has only trivial n.r.f., then the matrix

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & M_{n-r} \end{pmatrix},$$

which is in $P_n^{m\times n}$, can have only trivial n.r.f.

Proof. Let $A = BC$ be an n.r.f. of A. Partition

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \text{and} \quad C = (C_1, C_2),$$

where B_1 is $(m-n+r)\times n$ and C_1 is $n \times r$.

We may suppose, without loss of generality, that any zero columns of B_1 are at the right. Thus

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where B_{11} is $(m-n+r)\times k$, B_{12} is $(m-n+r)\times(n-k)$, B_{21} is $(n-r)\times k$, B_{22} is $(n-r)\times(n-k)$, C_{11} is $k\times r$, C_{21} is $(n-k)\times r$, C_{12} is $k\times(n-r)$, C_{22} is $(n-k)\times(n-r)$, $B_{12}=0$, and no column of B_{11} is 0. Clearly $k>0$, since A

has no zero row. We have

$$\begin{pmatrix} A_1 & 0 \\ 0 & M_{n-r} \end{pmatrix} = A = BC = \begin{pmatrix} B_{11}C_{11} & B_{11}C_{12} \\ B_{21}C_{11} + B_{22}C_{21} & B_{21}C_{12} + B_{22}C_{22} \end{pmatrix}$$

whence $0 = B_{11}C_{12}$.

Since no column of B_{11} is 0, it follows that $C_{12} = 0$. Hence $k < n$, since A has no zero column. Thus $0 < k < n$.

We now have $A_1 = B_{11}C_{11}$ and $M_{n-r} = B_{22}C_{22}$. We next show that $k = r$. If $k < r$, we have $A_1 = B'_{11}C'_{11}$, where

$$B'_{11} = (B_{11}, 0) \in P^{(m-n+r)\times r} \quad \text{and} \quad C'_{11} = \begin{pmatrix} C_{11} \\ 0 \end{pmatrix} \in P^{r\times r}.$$

But this factorization contradicts the fact that A_1 has only trivial n.r.f.

If $k > r$, we obtain $n - r < n - k$, a contradiction to $M_{n-r} = B_{22}C_{22}$ and the fact that M_{n-r} is a monomial. Hence $k = r$. But

$$A = BC = \begin{pmatrix} B_{11}C_{11} & 0 \\ B_{21}C_{11} + B_{22}C_{21} & B_{22}C_{22} \end{pmatrix},$$

and so $B_{21}C_{11} = 0$ and $B_{22}C_{21} = 0$. Since $A_1 = B_{11}C_{11}$ is an n.r.f. of A_1 and A_1 has only trivial n.r.f., it follows that C_{11} is a monomial. Thus C_{11} has no zero row. Hence it follows from $B_{21}C_{11} = 0$ that $B_{21} = 0$. Similarly, we deduce from $B_{22}C_{21} = 0$ and the fact that B_{22} is a monomial that $C_{21} = 0$. Hence

$$B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}, \qquad C = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix}$$

Since B_{22}, C_{22} are monomial and C_{11} is monomial, it follows that C is monomial. Thus A can have only trivial n.r.f. ∎

THEOREM 4.4. *Let $A \in P_n^{m\times n}$. Then $A \in P_{(1)}$ if and only if there exists an r, $1 \leqslant r \leqslant n$, and a fully indecomposable matrix $A_1 \in P_r^{(m-n+r)\times r}$ with $A_1 \in P_{(1)}$, such that*

$$PAQ = \begin{pmatrix} A_1 & 0 \\ 0 & D \end{pmatrix},$$

where P, Q are permutation matrices (of order m and n respectively) and D is a nonnegative nonsingular diagonal matrix of order $n - r$.

Proof. Immediate by Theorem 4.2 and Theorem 4.3. ∎

THEOREM 4.5. *Let $A \in P_n^{m\times n}$. If A has s ($\geqslant n$) positive rows and n of them are linearly independent, then A has infinitely many nontrivial n.r.f.'s of the form*

$$A = \begin{pmatrix} B_\alpha \\ FB_\alpha \end{pmatrix}(C_\alpha),$$

where B_α and C_α have been defined in the proof of Theorem 3.2 and F is an $(m-n)\times n$ matrix.

This conclusion can be easily derived from Theorem 3.2. The extreme vectors of the simplicial cone K which satisfies the strict cone-containment relation in Theorem 2.2 are the column vectors of the $m \times n$ matrix

$$\begin{pmatrix} B_\alpha \\ FB_\alpha \end{pmatrix}$$

THEOREM 4.6. *Let $A \in P_n^{m\times n}$ with $0 < r < \min\{m, n\}$. If there exist permutation matrices P, Q of suitable orders such that*

$$PAQ = \begin{pmatrix} M & MG \\ FM & FMG \end{pmatrix},$$

where M is an $r \times r$ nonsingular positive matrix, then $A \in P_{(2)}$. In fact, A has infinitely many (essentially different) nontrivial n.r.f.'s of the form

$$A = \begin{pmatrix} B_\alpha \\ FB_\alpha \end{pmatrix}(C_\alpha, C_\alpha M_r Z)$$

where B_α and C_α have been defined in the proof of Theorem 3.2, M_r is a monomial of order r, and z is a nonnegative matrix of order $r \times (n - r)$.

This conclusion can also be easily derived from Theorem 3.2.

5. SOME REMARKS

REMARK 1. Let $A \in P_n^{n \times n}$. A is said to have a nonnegative LU factorization if $A = LU$ with L a nonnegative lower triangular matrix and U a nonnegative upper triangular matrix. T. L. Markham [8] gave necessary and sufficient conditions for A to have a nonnegative LU factorization. That such an LU factorization is a nontrivial n.r.f. (except when A itself is a triangular matrix) is clear. We point out that whenever A is not a triangular matrix and $A = LU$ holds, then we have the strict cone-containment relation

$$AR_+^n \subset LR_+^n \subset \operatorname{Im} A \cap R_+^n = R_+^n,$$

and "almost" all the extreme vectors of the simplicial cone $K = LR_+^n$ are "lying" in the faces of the cone R_+^n.

REMARK 2. Similar definitions and arguments can be applied to the set $P_m^{m \times n}$ to obtain similar results.

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Note added in page proof. The author is thankful to Dr. B. S. Tam who correctly pointed out that Definition 1.2 should read as follows:

DEFINITION 1.2 Let $A \in P_n^{m \times n}$. If $A = BC$, where $B \in P_n^{m \times n}$, $C \in P_n^{n \times n}$ implies that either (1) $B = AM_n^{-1}$, $C = M_n$ for some monomial M_n, or (2) there exists permutation matrix P of order m such that

$$B = P\begin{pmatrix} M_n \\ 0 \end{pmatrix} \tag{1.3}$$

for some monomial M_n, then BC is called a trivial n.r.f. of A. Any $A \in P_n^{m \times n}$ can have a trivial n.r.f. If A has an n.r.f. $A = BC$ different from a trivial n.r.f., we will say that A has a nontrivial n.r.f.

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